Resolution of Singularities in Arbitrary Characteristic

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Let $X$ be an integral affine or projective scheme over a perfect field $k$ of an arbitrary characteristic. Then, $X/k$ admits a resolution. That is, there exists a smooth scheme $\tilde{X}/k$ and a projective birational morphism $\tilde{X} \rightarrow X$ onto $X$. 
Blow up the universe and resolve all singularities

Our approach:

- there exists a “universe” that contains arbitrary singularities, $Z_{\Gamma}$;
- in this universe, every singularity $Z_{\Gamma}$ is described in a “normal form”;
- we blow up the universe and produce a new universe;
- in this new universe, we look at the birational transforms $\tilde{Z}_{\Gamma}$ and prove that they all become smooth.

We never investigate the singularities of any individual singular variety $Z_{\Gamma}$, nor do we care how exactly $Z_{\Gamma}$ becomes smooth.

Such a universe is provided by Mnëv’s universality.
Mnëv’s universality

Any singularity over \( \mathbb{Z} \) appears in a thin Schubert cell of the Grassmannian \( \text{Gr}^{3,E} \) of three dimensional linear subspaces in a vector space \( E \).

The Plücker embedding \( \text{Gr}^{3,E} \subset \mathbb{P}(\wedge^3 E) \) with Plücker coordinates \( p_{ijk} \).

- Any Schubert divisor is defined by \( p_{ijk} = 0 \) for some \((ijk)\).
- A thin Schubert cell of \( \text{Gr}^{3,E} \) is an open subset of the closed subscheme of \( \text{Gr}^{3,E} \) defined by

\[
\{ p_{ijk} = 0 \mid p_{ijk} \in \Gamma \}
\]

for some subset \( \Gamma \) of all Plücker variables.

It must lie in an affine chart \((p_m \neq 0)\) for some \( m \in \mathbb{I}_{3,n} \). Thus, it is an open subset of the closed subscheme of \( \text{Gr}^{3,E} \cap (p_m \neq 0) \)

\[
Z_\Gamma = \{ p_{ijk} = 0 \mid p_{ijk} \in \Gamma \} \cap \text{Gr}^{3,E} \cap (p_m \neq 0).
\]

This is a closed affine subscheme of the affine space \((p_m \neq 0)\).

We aim to resolve \( Z_\Gamma \), hence the thin Schubert cell, for all \( \Gamma \) and all \( \text{Gr}^{d,E} \).
Up to permutation, we may assume that \( m = (123) \) and the chart is

\[
U_m := (p_{123} \neq 0).
\]

We write the de-homogenized coordinates of \( U_m \) as

\[
\{x_{abc} \mid (abc) \in \mathbb{I}_{3,n} \setminus \{(123)\} \}.
\]

As a closed subscheme of the affine space \( U_m \), \( Z_{\Gamma} \) is defined by

\[
\{x_{ijk} = 0, \quad \bar{F} = 0 \mid x_{ijk} \in \Gamma\},
\]

where \( \bar{F} \) runs over all de-homogenized Plücker relations. We need to pin down some explicit Plücker relations to form a minimal set of generators of \( \text{Gr}^{3,E} \cap U_m \).
Minimal set of Plücker relations for the chart \((p_m \neq 0)\).

They are of the following forms:

\[ \bar{F}_{(123),1uv} = x_{1uv} - x_{12}x_{13}v + x_{13}x_{12}v, \quad (1) \]
\[ \bar{F}_{(123),2uv} = x_{2uv} - x_{12}x_{23}v + x_{23}x_{12}v, \quad (2) \]
\[ \bar{F}_{(123),3uv} = x_{3uv} - x_{13}x_{23}v + x_{23}x_{13}v, \quad (3) \]
\[ \bar{F}_{(123),abc} = x_{abc} - x_{12}a x_{3bc} + x_{13}a x_{2bc} - x_{23}a x_{1bc}, \quad (4) \]

where \( u < v \in [n] \setminus \{1, 2, 3\} \) and \( a < b < c \in [n] \setminus \{1, 2, 3\} \). Here, \([n] = \{1, \ldots, n\}\).

Primary Plücker Relations

In a nutshell, we have the set

\[ \mathcal{F} := \mathcal{F}_m = \{ \bar{F}_{(123),iuv}, \ 1 \leq i \leq 3; \ \bar{F}_{(123),abc} \} \quad (5) \]

Every relation of \( \mathcal{F} \) is called \( m \)-primary. Here, \( m = (123) \). Every \( m \)-primary relation has a leading term.
Hence, as a closed subscheme of the affine space $\mathbb{U}_m$, $Z_\Gamma$ is defined by

$$Z_\Gamma = \{ xu = 0, \quad \bar{F}_{(123),iuv}, \quad 1 \leq i \leq 3, \quad \bar{F}_{(123),abc} \mid xu \in \Gamma \}. $$

This may be regarded as a normal form of the singularity type of $Z_\Gamma$. Upon setting $xu = 0$ with $u \in \Gamma$, we obtain the affine coordinate subspace

$$\mathbb{U}_{m,\Gamma} \subset \mathbb{U}_m$$

such that $Z_\Gamma$, as a closed subscheme of the affine subspace $\mathbb{U}_{m,\Gamma}$, is defined by

$$\mathcal{F}|_{\Gamma} := \{ \bar{F}_{(123),iuv}|_{\Gamma}, \quad 1 \leq i \leq 3; \quad \bar{F}_{(123),abc}|_{\Gamma} \}, \quad (6)$$

where $\bar{F}|_{\Gamma}$ denotes the restriction of $\bar{F}$ to the affine subspace $\mathbb{U}_{m,\Gamma}$. These are in general truncated Plücker equations, some of which may be identically zero. Note here that $\text{Gr}^{d,E} \cap (p_m \neq 0) = Z_0$. 

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Equations of $\Gamma$-schemes: normal forms of singularities

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We do not analyze singularities of $Z_{\Gamma}$.

But, we make a quick observation:
The singular scheme $Z_{\Gamma}$ is cut out from the affine chart $\mathbb{U}_m$ of the Grassmannian $\text{Gr}^{3, E}$ by the coordinate hyperplanes $x_u = 0$ for all $x_u \in \Gamma$. Although $Z_{\Gamma}$ (as well as the thin Schubert cell $Z_{\Gamma}^\circ$ in $Z_{\Gamma}$) is nicely described by Plücker variables and Plücker relations, the intersections of these coordinate hyperplanes with the chart $\mathbb{U}_m$ of the Grassmannian $\text{Gr}^{3, E}$ are arbitrary. We may view $\mathbb{U}_m$ (allowing $\text{Gr}^{3, E}$ to vary) as a universe that contains arbitrary singularities. Hence, intuitively, we need to birationally change the universe “along these intersections” so that eventually in the new universe, “they” re-intersect properly.

To achieve this, it is more workable if we can put all the singularities in a different universe $\mathbb{V}_m$, birationally modified from the chart $\mathbb{U}_m \cap \text{Gr}^{3, E}$, so that in the new model $\mathbb{V}_m$, all the terms of the above Plücker relations can be separated.
For simplicity, in what follows, we will write a general Plücker relation as

$$F = \sum_{s \in S_F} \text{sgn}(s) p_u s p_v s.$$ 

On the chart \((p_m = 1)\), upon setting \(x_m = 1\) and \(x_u = p_u, u \neq m\), we write the de-homogenization of \(F\) as

$$\bar{F} = \sum_{s \in S_F} \text{sgn}(s) x_u s x_v s.$$ 

The projective spaces \(\mathbb{P}_F\) and the linearized Plücker relations \(L_F\).

For later use, associated with \(F\) is the projective space \(\mathbb{P}_F\) with \([x(u_s, v_s)]_{s \in S_F}\) as its homogeneous coordinates. The linearized Plücker relation

$$L_F : \sum_{s \in S_F} \text{sgn}(s) x(u_s, v_s).$$
Blow up the universe and resolve all singularities

Here, $\text{Gr}^3, E$ or the chart $\text{Gr}^3, E \cap (\rho_m \neq 0)$ is the universe that contains all possible singularities, $Z_{\Gamma}$.

We blow up $\text{Gr}^3, E \cap (\rho_m \neq 0)$ to arrive at a new universe $\tilde{\mathcal{R}}_{\emptyset}$.

We then look at the birational transforms of $Z_{\Gamma}$ and prove that they all become smooth!

In stead of directly blowing up $\text{Gr}^3, E \cap (\rho_m \neq 0)$, we actually place all singularities $Z_{\Gamma}$ in a different universe $\mathcal{V}_m$ that is birational onto the chart $U_m \cap \text{Gr}^3, E$. We then blow up $\mathcal{V}_m$. 
Separating the terms of Plücker relations.

We construct a local model $V_m$, birational onto the chart $U_m \cap \text{Gr}^{3,E}$, such that in a specific set of defining binomial equations of $V_m$, all the terms of the above Plücker relations are separated/isolated.

There exists a rational map

$$
\Theta : U_m \rightarrow \prod_{\bar{F} \in \mathcal{F}m} \mathbb{P}_F,
$$

$$(x_u) \mapsto \prod_{\bar{F} \in \mathcal{F}m} [x_u_s x_v_s]_{s \in S_F}.$$

We then let $V_m$ be the closure of the graph of the rational map $\Theta|_{\text{Gr}^{3,E} \cap U_m}$. This is the birational model onto the chart $U_m \cap \text{Gr}^{3,E}$ and it is embedded in the smooth ambient scheme

$$
V_m \subset \mathcal{R}_F := U_m \times \prod_{\bar{F} \in \mathcal{F}m} \mathbb{P}_F.
$$
By calculating the multi-homogeneous kernel of the homomorphism
\[ \bar{\varphi} : k[(x_w); (x(u,v))] \longrightarrow k[x_w] \] (7)

we determine a set of defining relations of \( V_m \) in \( \mathcal{R}_F \). These defining relations, among many others, include the following binomials.

### The main binomial relations

- \( x_{1uv}x_{(12u,13v)} - x_{12u}x_{13v}x_{(123,1uv)} \), \( x_{1uv}x_{(13u,12v)} - x_{13u}x_{12v}x_{(123,1uv)} \),
- \( x_{2uv}x_{(12u,23v)} - x_{12u}x_{23v}x_{(123,2uv)} \), \( x_{2uv}x_{(23u,12v)} - x_{23u}x_{12v}x_{(123,2uv)} \),
- \( x_{3uv}x_{(13u,23v)} - x_{13u}x_{23v}x_{(123,3uv)} \), \( x_{3uv}x_{(23u,13v)} - x_{23u}x_{13v}x_{(123,3uv)} \),
- \( x_{abc}x_{(12a,3bc)} - x_{12a}x_{3bc}x_{(123,abc)} \), \( x_{abc}x_{(13a,2bc)} - x_{13a}x_{2bc}x_{(123,abc)} \), \( x_{abc}x_{(23a,1bc)} - x_{23a}x_{1bc}x_{(123,abc)} \).

The set of the main binomials is denoted \( B_{mn} \). For every \( B \in B_{mn} \),

\[ B = T_B^+ - T_B^- . \]
The defining relations of $\mathcal{V}_m$ in $\mathcal{R}_\mathcal{F}$

- the main binomial relations of (8);
- the residual binomial relations;
- the binomial relations of quotient type;
- the linearized Plücker relations $L_F, F \in \mathcal{F}_m$.

A residual binomial relation takes of the form, for example,

$$x_{12}ux_{13}v x_{(13u,12v)} - x_{13}u x_{12}v x_{(12u,13v)}, \quad x_{12}a x_{3bc} x_{(13a,2bc)} - x_{13}a x_{2bc} x_{(12a,3bc)}.$$  

(no leading variables.) The set of residual binomials is denoted $\mathcal{B}^{\text{res}}$.

A binomial relation of quotient type takes of the form, for example,

$$x_{(12a,13b)} x_{(13a,12c)} x_{(12b,13c)} - x_{(13a,12b)} x_{(12a,13c)} x_{(13b,12c)}.$$  

(no Plücker variables.) The set of binomials of quot type is denoted $\mathcal{B}^{\text{q}}$. 
In addition, for any primary Plücker relation

\[ F = \sum_{s \in S_F} \text{sgn}(s) p_{u_s} p_{v_s}, \]

it induces the linearized Plücker relation in \( \mathbb{P}_F \),

\[ L_F = \sum_{s \in S_F} \text{sgn}(s) x(u_s, v_s). \]

The set of all linearized Plücker relations is denoted \( L_{\mathcal{F}} \).
The closed subscheme $\mathcal{V}_m \subset \mathcal{R}_\mathcal{F} = \bigcup_m \times \prod_{\bar{F} \in \mathcal{F}_m} \mathbb{P}_F$ is defined by $\mathcal{B}^{mn}, \mathcal{B}^{\text{res}}, \mathcal{B}^{q}, L_\mathcal{F}$.

The smooth scheme $\mathcal{R}_\mathcal{F}$ comes equipped with the following divisors.

- we call $x_u$ (e.g., $x_{12u}$) a $\varpi$-variable and
  
  $$ X_u = (x_u = 0) $$

  a $\varpi$-divisor;

- we call $x_{(u,v)}$ (e.g., $x_{(12u,13v)}$) a $\varrho$-variable and
  
  $$ X_{(u,v)} = (x_{(u,v)} = 0) $$

  a $\varrho$-divisor.
The process of “removing” zero factors of main binomials.

The goal of “removing” zero factors from the main binomials of $B^{mn}$ is achieved through sequential blowups based upon factors of main binomials and their proper transforms. We break the sequential blowups into three types, named as $\vartheta$-, $\varphi$-, and $\partial$-blowups; besides “removing” zero factors, each serves its own corresponding purpose.

The set $\mathcal{F}_m$ comes equipped a natural ordering

$$\mathcal{F}_m = \{ \bar{F}_1 < \cdots < \bar{F}_\gamma \}.$$  

$\bar{F}_k$ has the leading variable $x_{u_k}$ for all $k \in [\gamma] = \{1, 2, \cdots, \gamma\}$.

The set $B^{mn}$ is equipped with a somewhat carefully chosen total ordering, compatible with the ordering on $\mathcal{F}_m$.

To remove zero factors of main binomials, we either work on each primary Plücker relation individually (in the case of a $\vartheta$-blowup), or work on each main binomial individually (in the case of a $\varphi$- or an $\partial$-blowup).
The main binomial relations

- \( x_{1uv}x_{(12u,13v)} - x_{12u}x_{13v}x_{(123,1uv)} \), \( x_{1uv}x_{(13u,12v)} - x_{13u}x_{12v}x_{(123,1uv)} \),
- \( x_{2uv}x_{(12u,23v)} - x_{12u}x_{23v}x_{(123,2uv)} \), \( x_{2uv}x_{(23u,12v)} - x_{23u}x_{12v}x_{(123,2uv)} \),
- \( x_{3uv}x_{(13u,23v)} - x_{13u}x_{23v}x_{(123,3uv)} \), \( x_{3uv}x_{(23u,13v)} - x_{23u}x_{13v}x_{(123,3uv)} \),
- \( x_{abc}x_{(12a,3bc)} - x_{12a}x_{3bc}x_{(123,abc)} \), \( x_{abc}x_{(13a,2bc)} - x_{13a}x_{2bc}x_{(123,abc)} \),
- \( x_{abc}x_{(23a,1bc)} - x_{23a}x_{1bc}x_{(123,abc)} \).

We still do not analyze singularities of the birational transform of \( Z_\Gamma \). But, we make a quick observation:

- When both terms of some of the main binomials vanish at a point, then a singularity is likely to occur.
- Thus, first, we want to “remove” all the zero factors of both terms of every of the the main binomial relations.
**On $\vartheta$-blowups.**

Besides removing certain zero factors, $\vartheta$-blowups also make (the proper transforms of) the residual binomial equations dependent on (the proper transforms of) the main binomial equations, on all standard charts.

From the main binomial equations of (8), we select the following closed centers

$$\mathcal{Z}_\vartheta : \quad (x_{iuv} = 0) \cap (x_{(123, iuv)} = 0), \ i \in [3];$$

$$\quad (x_{abc} = 0) \cap (x_{(123, abc)} = 0), \ a \neq b \neq c \in [n] \setminus [3].$$

It comes equipped with an induced ordering.

We then blow up $\mathcal{R}_F$ along (the proper transforms of) the centers in $\mathcal{Z}_\vartheta$, in the above order. This gives rise to the sequence

$$\tilde{\mathcal{R}}_\vartheta := \tilde{\mathcal{R}}_\vartheta[\Upsilon] \longrightarrow \cdots \longrightarrow \tilde{\mathcal{R}}_\vartheta[k] \longrightarrow \tilde{\mathcal{R}}_\vartheta[k-1] \longrightarrow \cdots \longrightarrow \tilde{\mathcal{R}}_\vartheta[0].$$
It induces the sequential blowups

\[ \tilde{V}_\vartheta = \tilde{V}_{\vartheta[\tau]} \longrightarrow \cdots \longrightarrow \tilde{V}_{\vartheta[k]} \longrightarrow \tilde{V}_{\vartheta[k-1]} \longrightarrow \cdots \longrightarrow \tilde{V}_{\vartheta[0]}, \]

where \( \tilde{V}_{\vartheta[k]} \subset \tilde{R}_{\vartheta[k]} \) is the proper transform of \( V_m \) in \( \tilde{R}_{\vartheta[k]} \), any \( k \in [\tau] \).

**Good properties of \( \vartheta \)-blowups**

- Upon completing \( \vartheta \)-blowups, we can discard all the residual binomials \( B_{\text{res}} \) from consideration.

- \( \tilde{V}_\vartheta \cap X_{\vartheta,(m,u_k)} = \emptyset \) for all \( k \in [\tau] \) where \( X_{\vartheta,(m,u_k)} \) is the proper transform of the \( \varrho \)-divisor \( X_{(m,u_k)} = (x_{(m,u_k)} = 0) \). In particular, the zero factors \( x_{(m,u_k)} \) are “removed”, for all \( k \in [\tau] \).

Here, \( u_k \) is the index for the leading term of \( \tilde{F}_k \) for all \( k \in [\tau] \).
**On ϕ-blowups.**

Here, we continue the process of “removing” zero factors of the proper transforms of the main binomials, focusing only on ϖ-variables and exceptional variables, but not any ϱ-variables.

We work on each main binomial $B \in \mathcal{B}^{mn}$, one by one, starting from the first one. Recall that $\mathcal{B}^{mn}$ is ordered.

The first main binomial equation of (8) is

$$B_{145} : x_{145}x_{(124,135)} - x_{124}x_{135}x_{(123,145)}.$$

For each and every term of $B_{145}$, we pick a “zero” factor to form a pair, but, we do not pick any ϱ-variable. We do not pick $x_{(123,145)}$ because $\tilde{\mathcal{V}}_{\varnothing} \cap X_{\varnothing,(123,145)} = \emptyset$; we do not pick $x_{(124,135)}$ for a good reason. Then, there are two such pairs, called ϕ-sets with respect to $B_{145}$.

$$\phi_1 = (x_{145}, x_{124}), \quad \phi_2 = (x_{145}, x_{135}).$$ (9)
They give rise to the $\wp$-centers

$$Z_{\phi_1} = X_{\vartheta,145} \cap X_{\vartheta,124}, \quad Z_{\phi_2} = X_{\vartheta,145} \cap X_{\vartheta,135},$$  \hspace{1cm} (10)

where $X_{\vartheta,u}$ is the proper transform of $X_u$ in $\tilde{R}_{\vartheta}$. We can then blow up $\tilde{R}_{\vartheta}$ along (the proper transforms) of $Z_{\phi_1}$ and $Z_{\phi_2}$.

We then move on to the next main binomial equation

$$B_{245} : x_{245}x(124,235) - x_{124}x_{235}x(123,245).$$

Note that the minus term of the proper transform of $B_{245}$ acquires the exceptional parameter $\zeta$ created by the blowup along $Z_{\phi_1}$ through the variable $x_{124}$ ($x_{124}$ either turns into the exceptional parameter $\zeta$ or acquires it). This is an additional “zero” factor in the proper transform of $B_{245}$. Then, for every term of the proper transform of $B_{245}$, we pick a factor, including exceptional parameters, to form a pair, and again, we do not pick any $\varphi$-variables. Such a pair is called a $\wp$-set with respect to $B_{245}$. They give rise to $\wp$-centers of co-dimension two in the previously obtained blowup scheme. We can blow up that scheme along (the proper transforms) of these centers.
We then move on to $B_{345}$, repeat the above, and so on. When all main binomials are exhausted, we obtain the sequential blowups

$$\tilde{R}_\varnothing \to \cdots \to \tilde{R}(\varnothing(k\tau)r_\mu s_h) \to \tilde{R}(\varnothing(k\tau)r_\mu s_{h-1}) \to \cdots \to \tilde{R}_\vartheta.$$ 

An intermediate blowup scheme in the above is denoted by $\tilde{R}(\varnothing(k\tau)r_\mu s_h)$. Here $(k\tau)$ is the index of a main binomial. As the process of $\varnothing$-blowups goes on, more and more exceptional parameters may be acquired and appear in the proper transform of the later main binomial $B_{(k\tau)}$, resulting more pairs of zero factors, hence more corresponding $\varnothing$-sets and $\varnothing$-centers. The existence of the index $r_\mu$, called *round* $\mu$, is due to the need to deal with the situation when an exceptional parameter with exponent greater than one is accumulated in the minus terms of the proper transform of the main binomial $B_{(k\tau)}$ (such a situation does not occur for the first few main binomials). The index $s_h$, called *step* $h$, simply indicates the corresponding step of the blowup.

Besides removing the zero factors, the reason that we exclude $\varnothing$-variables from $\varnothing$-sets is to control the binomial equations of quotient type.
Here, we finalize the process of “removing” zero factors of the proper transforms of all the main binomials. Like in the $\varphi$-blowups, we focus on each main binomial relation individually, starting from the first one. The construction is totally analogous to that of $\varphi$-blowup.

By induction, suppose we are now considering a main binomial $B(k\tau)$. For each and every term of the proper transform of the main binomial $B(k\tau)$ in the previously obtained blowup scheme, we pick a possible “zero” factor to form a pair. Here, we do not exclude any variable any more. Such a pair is called $\bar{\partial}$-sets with respect to $B(k\tau)$. They give rise to $\bar{\partial}$-centers with respect to $B(k\tau)$. The set of all $\bar{\partial}$-center comes equipped with a total order. We then blow up the previously obtained scheme along (the proper transforms) of these $\bar{\partial}$-centers.

This gives rise to the final sequential blowups

$$
\tilde{R}_{\bar{\partial}} \to \cdots \to \tilde{R}_{(\bar{\partial}_h\tau)} \to \tilde{R}_{(\bar{\partial}_h\tau)} \to \cdots \to \tilde{R}_{\varphi}.
$$
Although discussed in terms of coordinate variables, the constructions of all these \(\varphi\)- and \(\bar{\partial}\)-blowups, like \(\vartheta\)-blowups, are done globally via induction. They are all based upon main binomials:

\[
(\prod \varepsilon_i) x_{u_k} x_{u_s} x_{v_s} - (\prod \eta_j) x_{u_s} x_{v_s} x_{m, u_k}.
\]

- \(\tilde{R}_\vartheta := \tilde{R}_\vartheta[\tau] \to \cdots \to \tilde{R}_\vartheta[k] \to \tilde{R}_\vartheta[k-1] \to \cdots \to \tilde{R}_\vartheta[0],\)

\[
\tilde{V}_\vartheta = \tilde{V}_\vartheta[\tau] \to \cdots \to \tilde{V}_\vartheta[k] \to \tilde{V}_\vartheta[k-1] \to \cdots \to \tilde{V}_\vartheta[0].
\]

- \(\tilde{R}_\varphi \to \cdots \to \tilde{R}_{(\varphi(k\tau)r_\mu s_h)} \to \tilde{R}_{(\varphi(k\tau)r_\mu s_{h-1})} \to \cdots \to \tilde{R}_\vartheta,\)

\[
\tilde{V}_\varphi \to \cdots \to \tilde{V}_{(\varphi(k\tau)r_\mu s_h)} \to \tilde{V}_{(\varphi(k\tau)r_\mu s_{h-1})} \to \cdots \to \tilde{V}_\vartheta.
\]

- \(\tilde{R}_{\bar{\partial}} \to \cdots \to \tilde{R}_{(\bar{\partial}(k\tau)r_\mu s_h)} \to \tilde{R}_{(\bar{\partial}(k\tau)r_\mu s_{h-1})} \to \cdots \to \tilde{R}_\varphi,\)

\[
\tilde{V}_{\bar{\partial}} \to \cdots \to \tilde{V}_{(\bar{\partial}(k\tau)r_\mu s_h)} \to \tilde{V}_{(\bar{\partial}(k\tau)r_\mu s_{h-1})} \to \cdots \to \tilde{V}_\varphi.
\]
Fix any integral $\Gamma$-scheme $Z_\Gamma$, considered as a closed subscheme of $U_m \cap Gr^{3,E}$. Our goal is to resolve $Z_\Gamma$ when it is singular.

Recall that $V_m (\subset R_\mathcal{F})$ is birational onto $U_m \cap Gr^{3,E}$.

- $\exists Z_{\mathcal{F},\Gamma} \subset V_m$ with explicit defining equations such that it comes equipped with an irreducible components $Z^\dagger_{\mathcal{F},\Gamma}$ birational onto $Z_\Gamma$.

- $\exists \tilde{Z}_{\vartheta,\Gamma} \subset \tilde{V}_{\vartheta}$ with explicit defining equations such that it comes equipped with an irreducible components $\tilde{Z}^\dagger_{\vartheta,\Gamma}$ birational onto $Z^\dagger_{\mathcal{F},\Gamma}$.

- $\exists \tilde{Z}_{\varphi,\Gamma} \subset \tilde{V}_{\varphi}$ with explicit defining equations such that it comes equipped with an irreducible components $\tilde{Z}^\dagger_{\varphi,\Gamma}$ birational onto $\tilde{Z}^\dagger_{\vartheta,\Gamma}$.

- $\exists \tilde{Z}_{\partial,\Gamma} \subset \tilde{V}_{\partial}$ with explicit defining equations such that it comes equipped with an irreducible components $\tilde{Z}^\dagger_{\partial,\Gamma}$ birational onto $\tilde{Z}^\dagger_{\varphi,\Gamma}$.

Here, $Z_{\mathcal{F},\Gamma}$, $\tilde{Z}_{\vartheta,\Gamma}$, $\tilde{Z}_{\varphi,\Gamma}$, or $\tilde{Z}_{\partial,\Gamma}$ is either the proper transform of the previous one (via induction), or a birational slice of the total transform of the previous one.
Smoothness by Jacobian of main binomials and linearized Plücker relations over prime fields.

The question is local. So we focus on an arbitrary affine chart of $\mathcal{V}$ of $\mathcal{K}_0$. The chart $\mathcal{V}$ is isomorphic to an affine space and comes equipped with a set of (explicitly defined) coordinate variables $\text{Var}_\mathcal{V}$. The closed subscheme $\widetilde{Z}_{0,\Gamma} \cap \mathcal{V} \subset \mathcal{V}$ is defined by

$$B_{\mathcal{V}}^{mn|_{\Gamma_{\mathcal{V}}}}, L_{\mathcal{V},F m|_{\Gamma_{\mathcal{V}}}}, B_{\mathcal{V}}^{q|_{\Gamma_{\mathcal{V}}}}$$

for some $\Gamma_{\mathcal{V}} \subset \text{Var}_\mathcal{V}$ determined by $\Gamma$ and $\mathcal{V}$.

As envisioned, we confirm that the scheme $\tilde{\mathcal{V}}_0 (= \tilde{Z}_{0,\emptyset})$ is smooth on the chart $\mathcal{V}$ by some explicit calculations and careful analysis on the Jacobian of the main binomial relations of $B_{\mathcal{V}}^{mn}$ and linearized Plücker relations of $L_{\mathcal{V},F m}$. This implies that on the chart $\mathcal{V}$, the main binomial relations of $B_{\mathcal{V}}^{mn}$ and the linearized Plücker relations of $L_{\mathcal{V},F m}$ together generate the ideal of $\tilde{\mathcal{V}}_0 \cap \mathcal{V}$. Thus, as a consequence, the binomials of quotient type $B_{\mathcal{V}}^{q}$ can be discarded from consideration.
For any closed point $z \in \tilde{\mathcal{V}}$, we have

\[
\left( \begin{array}{cccc}
J^*(B_{F_1}^{\text{ori}}, L_{F_1}, B_{F_1}^{\text{inc}})(z) & 0 & \ldots & 0 \\
\ast & J^*(B_{F_2}^{\text{ori}}, L_{F_2}, B_{F_2}^{\text{inc}})(z) & \ldots & 0 \\
\ast & \ast & \ldots & \ast \\
\ast & \ast & \ast & J^*(B_{F_\Gamma}^{\text{ori}}, L_{F_\Gamma}, B_{F_\Gamma}^{\text{inc}})(z)
\end{array} \right)
\]

Then, the similar calculations and analysis on the Jacobian of the induced main binomial relations of $B_{F_\mathcal{V}}^{mn}|_{\tilde{\Gamma}_\mathcal{V}}$ and the induced linearized Plücker relations of $L_{\mathcal{V}}, F_m|_{\tilde{\Gamma}_\mathcal{V}}$ for $\tilde{Z}_{\partial, \Gamma}$ implies that $\tilde{Z}_{\partial, \Gamma}$ is smooth as well, on all charts $\mathcal{V}$. In particular, $\tilde{Z}_{\partial, \Gamma}^{\dagger}$, now a connected component of $\tilde{Z}_{\partial, \Gamma}$, is smooth, too.

This implies that $\tilde{Z}_{\partial, \Gamma}^{\dagger} \longrightarrow Z_{\Gamma}$ is a resolution, if $Z_{\Gamma}$ is singular.
Resolution Theorem.

Let $X$ be an integral affine or projective scheme over a perfect field $k$ of an arbitrary characteristic. Then, $X/k$ admits a resolution. That is, there exists a smooth scheme $\tilde{X}/k$ and a projective birational morphism $\tilde{X} \to X$ onto $X$.

Using Lafforgue’s version of Mnëv universality, we can apply the resolution $\tilde{Z}^{\dagger}_{0, \Gamma} \to Z_{\Gamma}$ to obtain a resolution for any singular integral affine scheme defined over $\mathbb{Z}$, and then for any singular integral projective scheme defined over $\mathbb{Z}$, using an affine cone over this projective scheme.

For a singular integral affine or projective scheme $X$ over general perfect field $k$, we can spread it out to obtain $f : Y \to B$ such that $X$ is isomorphic to the generic fiber of $f$ and $Y$ is defined over $\mathbb{Z}$. Take a resolution $\tilde{Y} \to Y$. Then, the generic fiber of $\tilde{Y} \to B$ provides a resolution of $X$. 
Blow up the universe and Resolve all singularities

- there exists a “universe” that contains arbitrary singularities, $Z_{\Gamma}$;
- we blow up the universe and produce a new universe;
- in this new universe, we look at the birational transforms $\tilde{Z}_{\Gamma}$ and prove that they all become smooth.